General Relativity

Final Exam
29/10/2015

Please write your first and last name and your student number on the first page.

Problem 1

Consider the 2-dimensional metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

defined for $y > 0$.

1. Write the components of the metric, compute the Christoffel symbols and the Ricci scalar (to compute the Ricci scalar, use the symmetries of the Riemann tensor).

2. Derive the geodesic equations for this metric.

3. Find the solution of the geodesic equations. Show that all geodesics begin and end on the line $y = 0$. Show that the total length of each geodesic is infinite. Find the shape of geodesics which intersect $y = 0$ at a right angle.

Problem 2

Consider a massive particle moving in the 2+1 dimensional spacetime with coordinates t, r, ϕ and metric

$$ds^2 = (r^2 - r_0^2)\, dt^2 - \frac{dr^2}{r^2 - r_0^2} - r^2 d\phi^2$$

The surface $r = r_0$ in this spacetime acts like a horizon of a black hole.

1. i) Derive the equations of motion of the particle and find the effective potential.
ii) Examine the possibility of circular orbits in this geometry

2. We release a particle with zero initial velocity from a point $r = R$. Show that the particle will fall radially but it will not cross $r = r_0$ in finite amount of coordinate time t. Calculate the amount of proper time needed for the particle to cross $r = r_0$.

3. An observer who is sitting at fixed value of $r = R$, releases a particle at $t = 0$. The particle starts falling towards the black hole. Suppose that after releasing the particle at $t = 0$, the observer remains fixed at $r = R$ until the time $t = T$. Right after $t = T$, he decides to dive in and to try and recover the infalling particle before it crosses the horizon. What is the minimum value of T, for which it is certainly impossible for the observer to recover the particle before it enters the horizon? Notice, it is ok to leave your final answer in terms of a definite integral (which is a little hard to evaluate explicitly).

Problem 3

Consider a Robertson-Walker metric with flat spatial section ($k = 0$):

$$ds^2 = dt^2 - a^2(t) \left[dx^2 + dy^2 + dz^2\right]$$

1. Check that particles staying at $x, y, z =$ constant follow geodesic motion.

2. Suppose that the universe is filled with pressure-less matter. Solve the FRW equations and find $a(t)$. Do the same computation for radiation dominated universe.

3. Suppose that in a matter dominated universe the big bang took place at time $t = 0$. Consider two particles at time $t = T$ (today). How far in space do they have to be, so that these particles could not have had any causal contact in the past? Give your answer in physical spacelike distance at time $t = T$.

Solution problem 1:

1. We have

$$g_{xx} = g_{yy} = \frac{1}{y^2} \qquad g_{xy} = g_{yx} = 0$$

The nonzero Christoffel symbols are

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$$

$$\Gamma^y_{xx} = \frac{1}{y}$$

$$\Gamma^y_{yy} = -\frac{1}{y}$$

First we compute

$$R^x_{yxy} = \partial_x \Gamma^x_{yy} - \partial_y \Gamma^x_{yx} + \Gamma^e_{yy}\Gamma^x_{ex} - \Gamma^e_{yx}\Gamma^x_{ey} = -\frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{y^2} = -\frac{1}{y^2}$$

$$R_{xyxy} = g_{xx} R^x_{yxy} = -\frac{1}{y^4}$$

The Ricci scalar is defined (in the conventions of Foster+Nightingale) by

$$R = g^{ad} g^{bc} R_{abcd} = y^4 (R_{xxxx} + R_{xyyx} + R_{yxxy} + R_{yyyy}) = 2$$

2. Using the Christoffel symbols we have

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0$$

$$\ddot{y} - \frac{1}{y}\dot{y}^2 + \frac{1}{y}\dot{x}^2 = 0$$

The proper-length parametrization implies $\frac{\dot{x}^2 + \dot{y}^2}{y^2} = 1$.

3. The first equation can be integrated by multiplying it with $\frac{1}{y^2}$. It then become

$$\frac{d}{d\tau}\left(\frac{\dot{x}}{y^2}\right) = 0$$

From this we learn $\dot{x} = ky^2$ for some constant k. From the proper time parametrization we have $\dot{y}^2 = y^2 - k^2 y^4$. This gives

$$\dot{y} = y\sqrt{1 - k^2 y^2}$$

or

$$\frac{dy}{y\sqrt{1 - k^2 y^2}} = d\tau$$

We define $y = \frac{1}{k}\sin\theta$ and we find

$$\frac{d\theta}{\sin\theta} = d\tau$$

$$\tan\frac{\theta}{2} = ce^{\tau}$$

or after some trigonometric identities

$$\sin\theta = \frac{2ce^{\tau}}{1 + c^2 e^{2\tau}}$$

or

$$y = \frac{1}{k}\frac{2ce^{\tau}}{1 + c^2 e^{2\tau}}$$

Select the origin of τ so that $c = 1$. Then we find

$$y = \frac{1}{k}\frac{2e^{\tau}}{1 + e^{2\tau}}$$

For $\tau = 0$ we find $y_{max} = 1/k$. For $\tau \to \mp\infty$ we find $y \to 0$.

We go to the equation for x and we have

$$\dot{x} = ky^2 = \frac{1}{k}\frac{4e^{2\tau}}{(1 + e^{2\tau})^2}$$

or

$$dx = \frac{1}{k}\frac{4e^{2\tau}}{(1 + e^{2\tau})^2} d\tau$$

we set

$$e^{2\tau} + 1 = s$$

$$dx = \frac{1}{k}\frac{2ds}{s^2}$$

$$x = \text{constant} - \frac{1}{k}\frac{2}{s}$$

$$x = \text{constant} - \frac{2}{k}\frac{1}{e^{2\tau} + 1}$$

Redefine constant so that

$$x = x_0 + \frac{1}{k} - \frac{2}{k}\frac{1}{e^{2\tau} + 1} = x_0 + \frac{1}{k}(1 - \frac{2}{e^{2\tau} + 1}) = x_0 + \frac{1}{k}\frac{e^{2\tau} - 1}{e^{2\tau} + 1}$$

We notice that

$$(x - x_0)^2 + y^2 = \frac{1}{k^2}\left[\frac{4e^{2\tau} + (e^{2\tau} - 1)^2}{(1 + e^{2\tau})^2}\right] = \frac{1}{k^2}$$

Since $y > 0$ this is the equation for a semicircle or radius $1/k$, whose center is at $(x_0, 0)$.

Solution problem 2:

1.

$$\dot{t}(r^2 - r_0^2) = k$$

$$r^2\dot{\phi} = h$$

$$\dot{t}^2(r^2 - r_0^2) - \dot{r}^2(r^2 - r_0^2)^{-1} - r^2\dot{\phi}^2 = 1$$

The last equation can be rewritten as

$$\frac{k^2}{r^2 - r_0^2} - \frac{\dot{r}^2}{r^2 - r_0^2} - \frac{h^2}{r^2} = 1$$

or

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left[(1 + \frac{h^2}{r^2})(r^2 - r_0^2)\right] = k^2$$

or

$$V(r) = \frac{1}{2}(1 + \frac{h^2}{r^2})(r^2 - r_0^2)$$

ii). Extremizing the potential we find the condition

$$2r + \frac{2h^2 r_0^2}{r^3} = 0$$

which is impossible to satisfy. Hence there are no circular orbits.

2. We notive that $h = 0$. Then we consider

$$\frac{k^2}{r^2 - r_0^2} - \frac{\dot{r}^2}{r^2 - r_0^2} = 1$$

at the initial condition we have $\dot{r} = 0$ hence

$$k = \sqrt{R^2 - r_0^2}$$

Plugging it back in we find

$$R^2 - r_0^2 - \dot{r}^2 = r^2 - r_0^2$$

or

$$\dot{r}^2 + r^2 = R^2$$

The solution is

$$r(\tau) = R\cos\tau$$

hence in finite amount of proper time it reaches $r = r_0$ as

$$r_0 = R\cos(\tau) \to \tau = \arccos(r_0/R)$$

On the other hand, using $\dot{t}(r^2 - r_0^2) = k$ we find

$$\frac{dt}{dr} = -\frac{k}{r^2 - r_0^2}\frac{1}{\sqrt{R^2 - r^2}}$$

or

$$t = -\int_R^{r_0} dr \frac{k}{r^2 - r_0^2}\frac{1}{\sqrt{R^2 - r^2}}$$

From this we notice that the integral is log-divergent.

3. In order to determine the minimum T we need to find the smallest T for which a light ray emitted at $t = T$ barely manages to get to the particle before the horizon. To proceed we need to find that orbit of the infalling particle. This we found before

$$t(r) = \int_r^R dr' \frac{\sqrt{R^2 - r_0^2}}{r'^2 - r_0^2}\frac{1}{\sqrt{R^2 - r'^2}}$$

The light ray obeys

$$(r^2 - r_0^2)dt = dr$$

or

$$t(r) = \int_r^R \frac{1}{(r^2 - r_0^2)} + T$$

Let's say they meet at $r = r_*$. Then we have

$$\int_{r_*}^R dr \frac{1}{(r^2 - r_0^2)} + T = \int_{r*}^R dr \frac{\sqrt{R^2 - r_0^2}}{r^2 - r_0^2}\frac{1}{\sqrt{R^2 - r^2}}$$

or

$$T = \int_{r^*}^{R} dr \frac{1}{r^2 - r_0^2} \left[\frac{\sqrt{R^2 - r_0^2}}{\sqrt{R^2 - r^2}} - 1 \right]$$

or the maximum T is when $r^* = r_0$. We have

$$T = \int_{r_0}^{R} dr \frac{1}{r^2 - r_0^2} \left[\frac{\sqrt{R^2 - r_0^2}}{\sqrt{R^2 - r^2}} - 1 \right]$$

Notice that the integral is convergent near the boundary $r = r_0$.

Solution problem 3:

2. For matter dominated we have $\rho a^3 =$ constant hence $\rho = \frac{A}{a^3}$. Pugging into FRW we find

$$\frac{\dot{a}^2}{a^2} = \frac{A}{a^3}$$

or

$$\sqrt{a} da = A dt$$

or

$$a(t) = A' t^{2/3}$$

3. The equation for a light ray is

$$dt - a(t) dx = 0$$

or

$$dx = \frac{dt}{a(t)}$$

or

$$x_{\max} = \int_0^T \frac{dt}{a(t)} = \frac{1}{A'} \int_0^T t^{-2/3} = \frac{3}{A'} T^{1/3}$$

Physical distance today

$$2a(T) x_{\max} = 2A' T^{2/3} \frac{3}{A'} T^{1/3} = 6T$$